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Hermite Wavelets Collocation Method for Solving Nonlinear Fredholm Integral Equations

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Abstract

In this research, we present Hermite wavelet collocation method (HeWM) for solving nonlinear Fredholm integral equations. The proposed method is based on operational matrices of Hermite wavelets, Leibnitz rule of integration and collocation points. Some numerical experiments have been performed to illustrate the accuracy and efficiency of the proposed method.

Keywords: Hermite wavelets, Leibnitz rule of integration, Operational matrices, Numerical examples.

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Introduction

Integral equations are encountered in many applications of science and engineering such as potential theory, acoustics, geophysics, renewal theory, electricity and magnetism, elasticity, fluid mechanics, mathematical problems of radiative equilibrium, mathematical economics and theory of population. So due to the challenges being faced by many researchers, there is an acute requirement to emphasize the importance of interdisciplinary effort and computational approach to advance the study of solving integral equations for further scientific research. In the past literature, many research papers have been published to present and establish different numerical methods for solving integral equations. Besides some iterative methods like Homotopy perturbation method (HPM), Adomian decomposition method (ADM), and Variational Iteration Method (VIM), various conventional methods such as Fourier spectral method, Galerkin method, collocation method, finite element method and finite difference method have been mentioned and used to solve linear and nonlinear integral equations. All these numerical schemes have been successfully applied for solving many integral equations which are one of the essential tools for various areas of applied mathematics and many other fields, including continuum mechanics, kinetic theory of gases, hereditary phenomena in physics and biology, quantum mechanics, radiation, optimization, optimal control systems, communication theory, queuing theory, medicine, the particle transport problems of astrophysics and reactor theory and the steady-state heat conduction. But due to some shortcomings of these numerical methods, researchers are making efforts to find more efficient alternatives for obtaining solutions to many practical and physical problems giving rise to integral equations.

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Wavelets theory is a relatively new and emerging area in mathematical research and is being extensively used as a powerful tool in various science and engineering disciplines. Wavelets are mathematical functions that have been widely used in digital signal processing for waveform representation and segmentation, image compression, time-frequency analysis, quick algorithms for easy implementations and many other fields of pure and applied mathematics. In the recent years, the different types of wavelet methods have found their way for the numerical solution of different kinds of integral equations arising in mathematical physics models and many other scientific and engineering problems. In, Maleknejad and Sahlan,[1] linear semi-orthogonal compactly supported spine wavelets as basis functions have been used to propose an advanced numerical model for the efficient solution of linear Fredholm Integral equation of the second kind. Along with an elementary introduction to the basic and main concepts of variational iteration Method, this method's theory and applications and the most new recent results have been put together in a systematic and convenient form He.[2] Block-Pulse functions have been formulated in Maleknejad *et al*.[3] to demonstrate the solution for the Fredholm integral equations system of the second kind.

In Maleknejad & Mirzaee,[4] the operational matrix of the product of rationalized Haar functions vector is utilized to reduce the computation of linear Fredholm integral equations. In Ray & Sahu,[5] different numerical methods have been examined for solving both linear and nonlinear Fredholm integral equations of second kind.Variational iteration method has been successfully applied to find the approximate solution of Fredholm integral equation of both linear and nonlinear types. In Singh and Kumar,[6] an efficient Haar wavelet method has been proposed for the numerical solution of a class of nonlinear Volterra integral equations of the first kind by converting them into linear Volterra integral equations of the second kind. Numerical examples have been given to illustrate that this method gives better results than the numerical methods described in past. The numerical solution of nonlinear Fredholm integral equations using Leibnitz-Haar wavelet collocation method has been obtained in Shiralashettiand and Mundewadi.[7] With the help of Leibnitz rule, the integral equations are converted into differential equations with initial conditions and thereafter the Haar wavelet function and its operational matrix are employed to solve the resulting differential equations. In Mirzaee and Samadyar,[8] a new method based on operational matrices of Bernoulli wavelet has been developed for solving linear stochastic Itô-Volterra integral equations. By applying these matrices, the main problem is transformed into a linear system of algebraic equations, which can be solved by using a suitable numerical method. In Heydari *et al.*,[9] a new computational method based on the Chebyshev wavelets (CW) is proposed for solving nonlinear stochastic Itô–Volterra integral equations.

Integral equations have attracted attention for most of the last century and their theory is continuously developing. The potential theory contributed more than any field to give rise to nonlinear integral equations. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, and water waves also contributed to the creation of nonlinear integral equations. The nonlinearity of these models may give more than one solution and this is the nature of nonlinear problems. Fredholm integral equation is one of the most important integral equation with enormous application in the theory of signal processing, linear forward modelling, inverse problems and in fluid mechanics problems involving hydrodynamic interactions near finite-sized elastic interfaces. In this paper, we have made an endeavour to develop Hermite wavelet based numerical technique using collocation points and Leibnitz rule in order to solve nonlinear Fredholm integral equations.

Leibnitz's Rule[10]

Leibnitz's rule of integration states that for any integral of the form

$$\int_{c(x)}^{d(x)} F(x,t)\,dt,$$

where $-\infty<c(x),d(x)<\infty$. The derivative of this integral is expressible as

$$\frac{d}{dx}\left[\int_{c(x)}^{d(x)} F(x,t)\,dt\right] = F\big(x,d(x)\big).\frac{d}{dx}d(x)$$

$$-F\big(x,c(x)\big).\frac{d}{dx}c(x) + \int_{c(x)}^{d(x)} \frac{\partial}{\partial x}F(x,t)\,dt,$$

where the partial derivative indicates that inside the integral, only the variation of $F(x, t)$ with x is considered in taking the derivative. If $a(x)$ and $b(x)$ are constants rather than function of x, then

$$\frac{d}{dx}\left(\int_c^d F(x,t)\,dt\right) = \int_c^d \frac{\partial}{\partial x}F(x,t)\,dt.$$

Hermite Wavelets and their Properties

Wavelets constitute a family of mathematical functions $\psi_{a,b}$ derived from dilation (change of scale) and translation (change of position) of a single function ψ called the mother wavelet. If the dilation parameter 'a'and translation parameter 'b' are considered to vary continuously, the family of continuous wavelets can be written as

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right),\ a>0,\ b\in R$$

By restricting the parameters a and b to discrete values as

$$a = a_0^{-k},\ b = nb_0a_0^{-k},\ a_0>1,\ b_0>0,$$

we obtain the following family of discrete wavelets:

$$\psi_{k,n}(x) = |a|^{-1/2}\psi(a_0^k x - nb_0),\ \forall a,b\in R,\ a\neq 0,$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$.

In particular, when we choose $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ form an orthonormal basis. Hermite wavelets are defined as

$$\psi_{n,m}(x) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}H_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}} \\ 0 & \text{, otherwise} \end{cases}$$

where $m = 0,1\cdots,M-1$, and $H_m(x)$ is Hermite polynomial of degree m.

Hermitepolynomials $H_n(x)$, are the solutions of Hermite's differential equation given by

$$y'' - 2xy' + 2xy = 0, \quad n = 0,1,2,3\ldots\ldots$$

These polynomials are given by the Rodrigue's formula

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

and are defined in the interval $(-\infty,\infty)$.

Operational matrices of Integration (OMI)

For operational matrix of integration for Hermite wavelets, we take $k=1$ and $M = 6\ (m = 0,1,2,3,4,5)$. The six basis functions on $[0,1)$ are given by

$$\psi_{1,0}(x) = \frac{2}{\sqrt{\pi}},$$

$$\psi_{1,1}(x) = \frac{2}{\sqrt{\pi}}(4x-2),$$

$$\psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(16x^2-16x+2),$$

$$\psi_{1,3}(x) = \frac{2}{\sqrt{\pi}}(64x^3-96x^2+24x+4),$$

$$\psi_{1,4}(x) = \frac{2}{\sqrt{\pi}}(256x^4-512x^3+192x^2+64x-20),$$

$$\psi_{1,5}(x) = \frac{2}{\sqrt{\pi}}(1024x^5-2560x^4+1280x^3+640x^2-400x+8).$$

Let

$\psi_6(x) = (\psi_{1,0}(x), \psi_{1,1}(x), \psi_{1,2}(x), \psi_{1,3}(x), \psi_{1,4}(x), \psi_{1,5}(x))^T$

Integrating the above basis functions with respect to x from 0 to x and expressing in matrix form, we obtain

$$\int_0^x \psi_{1,0}(x)dx = \frac{2}{\sqrt{\pi}}x = \left[\frac{1}{2},\frac{1}{4},0,0,0,0\right]\psi_6(x)$$

$$\int_0^x \psi_{1,1}(x)dx = \frac{2}{\sqrt{\pi}}(2x^2-2x) = \left[\frac{-1}{4},0,\frac{1}{8},0,0,0\right]\psi_6(x)$$

$$\int_0^x \psi_{1,2}(x)dx = \frac{2}{\sqrt{\pi}}\left(\frac{16}{3}x^3-8x^2+2x\right) = \left[\frac{-1}{3},0,0,\frac{1}{12},0,0\right]\psi_6(x)$$

$$\int_0^x \psi_{1,3}(x)dx = \frac{2}{\sqrt{\pi}}(16x^4-32x^3+12x^2+4x) = \left[\frac{-5}{4},0,0,0,\frac{1}{16},0\right]\psi_6(x)$$

$$\int_0^x \psi_{1,4}(x)dx = \frac{2}{\sqrt{\pi}}\left(\frac{256}{5}x^5-128x^4+64x^3+32x^2-20x\right) = \left[\frac{-2}{5},0,0,0,0,\frac{1}{20}\right]\psi_6(x)$$

$$\int_0^x \psi_{1,5}(x)dx = \frac{2}{\sqrt{\pi}}\left(\frac{512}{3}x^6-512x^5+560x^4-\frac{800}{3}x^3+50x^2-2x\right)$$

$$= \left[\frac{-23}{3},0,0,0,0,0\right]\psi_6(x) + \frac{1}{24}\psi_{1,6}(x)$$

Therefore,

$$\int_0^x \psi_6(x)dx = P_{6\times 6}\psi_6(x) + \bar{\psi}_6(x)$$

where

$$P_{6\times6} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{-1}{4} & 0 & \frac{1}{8} & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & 0 & \frac{1}{12} & 0 & 0 \\ \frac{-5}{4} & 0 & 0 & 0 & \frac{1}{16} & 0 \\ \frac{-2}{5} & 0 & 0 & 0 & 0 & \frac{1}{20} \\ \frac{-23}{3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\bar{\psi}_6(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{24}\psi_{1,6}(x) \end{bmatrix}$$

The double integration of above six basis is given by,

$$\int_0^x\int_0^x \psi_{1,0}(x)dx\,dx = \frac{2}{\sqrt{\pi}}\frac{x^2}{2} = \left[\frac{3}{16},\frac{1}{8},\frac{1}{32},0,0,0\right]\psi_6(x)$$

$$\int_0^x\int_0^x \psi_{1,1}(x)dx\,dx = \frac{2}{\sqrt{\pi}}\left(\frac{2}{3}x^3-x^2\right) = \left[\frac{-1}{6},\frac{-1}{16},0,\frac{1}{96},0,0,\right]\psi_6(x)$$

$$\int_0^x\int_0^x \psi_{1,2}(x)dx\,dx = \frac{2}{\sqrt{\pi}}\left(\frac{4}{3}x^4-\frac{8}{3}x^3+x^2\right) = \left[\frac{-1}{16},\frac{-1}{12},0,0,\frac{1}{192},0\right]\psi_6(x)$$

$$\int_0^x\int_0^x \psi_{1,3}(x)dx\,dx = \frac{2}{\sqrt{\pi}}\left(\frac{16}{5}x^5-8x^4+4x^3+2x^2\right) = \left[\frac{3}{5},\frac{5}{16},0,0,0,\frac{1}{320},0\right]\psi_6(x)$$

$$\int_0^x\int_0^x \psi_{1,4}(x)dx\,dx = \frac{2}{\sqrt{\pi}}\left(\frac{128}{15}x^6-\frac{128}{5}x^5+16x^4+\frac{32}{3}x^3-10x^2\right)$$

$$= \left[\frac{-7}{12},\frac{-1}{10},0,0,0,0\right]\psi_6(x) + \frac{1}{480}\psi_{1,6}(x)$$

$$\int_0^x\int_0^x \psi_{1,5}(x)dx\,dx = \frac{2}{\sqrt{\pi}}\left(\frac{512}{21}x^7-\frac{256}{3}x^6+64x^5+\frac{160}{3}x^4-\frac{200}{3}x^3+4x^2\right)$$

$$= \left[\frac{-22}{7},\frac{-23}{12},0,0,0,0\right]\psi_6(x) + \frac{1}{672}\psi_{1,7}(x)$$

Hence

$$\int_0^x\int_0^x \psi_6(x)dx\,dx = P'_{6\times6}\psi_6(x) + \bar{\psi}'_6(x)$$

where

$$P'_{6\times6} = \begin{bmatrix} \frac{3}{16} & \frac{1}{8} & \frac{1}{32} & 0 & 0 & 0 \\ \frac{-1}{6} & \frac{-1}{16} & 0 & \frac{1}{96} & 0 & 0 \\ \frac{-1}{16} & \frac{-1}{12} & 0 & 0 & \frac{1}{192} & 0 \\ \frac{3}{5} & \frac{5}{16} & 0 & 0 & 0 & \frac{1}{320} \\ \frac{-7}{12} & \frac{-1}{10} & 0 & 0 & 0 & 0 \\ \frac{-22}{7} & \frac{-23}{12} & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\bar{\psi}'_6(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{480}\psi_{1,6}(x) \\ \frac{1}{672}\psi_{1,7}(x) \end{bmatrix}$$

Similarly we can take any number of basis functions to find the corresponding operational matrices of integration.

Function Approximation

We would like to bring a solution function under Hermite space by approximating by elements of Hermite wavelet basis as follows,

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x)$$

To approximate $y(x)$, by truncating this series, we get

$$y(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\psi\,(x)$$

where C and ψ (x) are $2^{k-1}M \times 1$ matrix,

$$C^T = [C_{1,0},\ldots C_{1,M-1}, C_{2,0},\ldots C_{2,M-1},\ldots C_{2^{k-1},0},\ldots C_{2^{k-1},M-1}]$$

$$\psi(x) = [\psi_{1,0},\ldots \psi_{1,M-1}, \psi_{2,0},\ldots, \psi_{2,M-1},\ldots \psi_{2^{k-1},0},\ldots \psi_{2^{k-1},M-1}]^T$$

Convergence analysis

Theorem: A continuous function *y(x)* in H^2 *[0,1)* defined on *[0, 1)* be bounded, then the Hermite wavelets expansion of *y(x)* converges to it.

Proof: Let $y(x)$ be a bounded real valued function on $[0,1)$. The Hermite coefficients of continuous functions $y(x)$ is defined as,

$$C_{n,m} = \int_0^1 y(x)\psi_{n,m}\,dx = \int_I y(x)\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}H_m(2^k x - 2n + 1)\,dx,$$

where $I = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$.

Substituting, we obtain $2^k x - 2n + 1 = z$

$$C_{n,m} = \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\int_{-1}^{1} y\left(\frac{z-1+2n}{2^k}\right)H_m(z)2^{-k}\,dz$$

$$= \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}}\int_{-1}^{1} y\left(\frac{z-1+2n}{2^k}\right)H_m(z)\,dz$$

Using GMVT for integrals,

$$C_{n,m} = \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}}y\left(\frac{w-1+2n}{2^k}\right)\int_{-1}^{1} H_m(z)\,dz,$$

for some $w \in (-1,1)$

Put $\int_{-1}^{1} H_m(z)\,dz = h$,

$$|C_{n,m}| = \left|\frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}}\right| y\left|\frac{w-1+2n}{2^k}\right| h$$

Since y is bounded, therefore, $\sum_{n,m=0}^{\infty} c_{n,m}$ is absolutely convergent. Hence the Hermite series expansion of $y(x)$ converges uniformly.

Proposed Hermite wavelet collocation method (HeWM)

Consider nonlinear Fredholm integral equation of the second kind

$$u(t) = f(t) + \int_0^1 K(t,s)\,u(s)ds \qquad (1)$$

where $K(t,s)$ is a nonlinear function defined on $[0,1] \times [0,1]$ and is known as the kernel of the integral equation. The unknown function $u(t)$ represents the solution of the integral equation. With the help of Leibnitz rule of integration, we convert the above integral equation into an equivalent differential equation

Let

$$F(t) = \int_0^1 K(t,s)u(s)ds, \qquad (2)$$

Using Leibnitz rule of integration, we obtain

$$F'(t) = \frac{dF}{dt} = \int_0^1 \frac{\partial K(t,s)}{\partial t}u(s)ds \qquad (3)$$

Differentiating twice w.r.t. and using Leibnitz rule, we get,

$$u'(t) = f'(t) + F'(t) \qquad (4)$$

$$u''(t) = f''(t) + F''(t) \qquad (5)$$

withinitial conditions

$$u(0) = \beta, u'(0) = \gamma \qquad (6)$$

Consider the approximation

$$u''(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(t) \qquad (7)$$

Integrating twice and using , we get

$$u'(t) = \gamma + \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}P_{n,m}(t) \qquad (8)$$

$$u(t) = \beta + \gamma x + \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}P'_{n,m}(t) \qquad (9)$$

Substituting (7) – (9) in the differential equation (5), we obtain a nonlinear system of equations. From here, Hermite wavelet coefficients are obtained with the help of Newton's method and substituting the values of these coefficients in (9), we get the required approximate solutions of equation (1).

Numerical examples

In this section, we present some numerical examples to demonstrate the applicability and suitability of the above method.

Example 1:

Solve

$$u(t) + \int_0^1 e^{t-2s}[u(s)]^3 ds = e^{t+1}, \quad 0 \le t \le 1 \qquad (10)$$

with initial conditions $u(0) = 1$ whose exact solution is $u(t) = e^t$.

Differentiating (10) w.r.t. and using Leibniz rule, the equivalent differential equation is

$$u'(t) = e^{t+1} - \int_0^1 e^{t-2s}[u(s)]^3 ds \qquad (11)$$

That is

$$u'(t) - u(t) = 0 \qquad (12)$$

Assume that,

$$u'(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\Psi_{n,m}(t) \qquad (13)$$

Integrating (13), we obtain

$$u(t) = 1 + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}P_{n,m}(t) \qquad (14)$$

Substituting (13)-(14) in the differential equation (12), we obtain the following the system of equations

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\Psi_{n,m}(t) - \left(1 + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}P_{n,m}(t)\right) = 0 \qquad (15)$$

From here, Hermite wavelet coefficients are obtained. Approximate solution is obtained form (14) by substituting the values of wavelet coefficients into (14).

Table 1 shows the comparison of solutions and maximum absolute errors of Example 1. Figure 1 shows the comparison of exact and Hermite wavelet solutions of Example 1.

Example 2:

Consider the nonlinear Fredholm integral equation

$$u(t) - \int_0^1 ts[u(s)]^3 dt = e^t - \frac{(1+2e^3)t}{9}, \quad 0 \le t \le 1 \qquad (16)$$

with initial conditions and exact $u(0) = 1, u'(0) = 1$ solution as $u(t) = e^t$.

Differentiating (16) w.r.t and using Leibniz rule of integration, we obtain the differential equation

$$u'(t) = e^t - \frac{(1+2e^3)t}{9} + \int_0^1 s[u(s)]^3 dt \qquad (17)$$

$$u''(t) - e^t = 0 \qquad (18)$$

Consider an approximation

$$u''(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\Psi_{n,m}(t) \qquad (19)$$

Integrating (19) twice and using initial conditions, we obtain

$$u'(t) = 1 + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}P_{n,m}(t) \qquad (20)$$

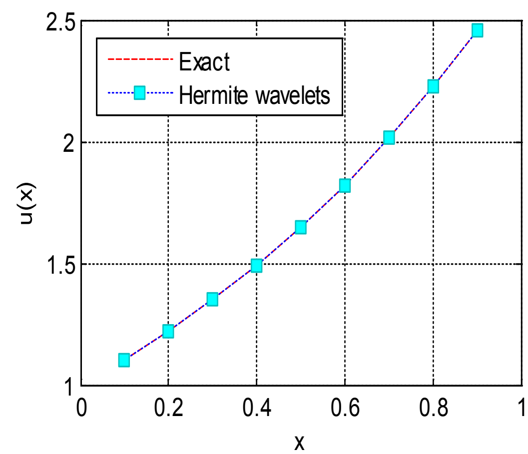

Figure 1: Comparison of exact and Hermite wavelet solution

Table 1: Maximum error analysis of Example 1

	Exact solution	*HeWM (k = 1, M = 6)*	*Absolute errors of HeWM*	*Absolute errors in [7]*	*Absolute errors in [11]*
0.1	1.1051709180	1.1051706507	2.6E-07	1.4E-04	2.2E-03
0.2	1.2214027581	1.2214025365	2.2E-07	1.6E-04	3.3E-03
0.3	1.3498588075	1.3498585670	2.4E-07	1.9E-04	8.7E-03
0.4	1.4918246976	1.4918244113	2.8E-07	2.3E-04	1.6E-02
0.5	1.6487212707	1.6487209634	3.0E-07	2.6E-04	1.8E-02
0.6	1.8221188003	1.8221184703	3.3E-07	3.1E-04	1.1E-02
0.7	2.0137527074	2.0137523226	3.8E-07	3.6E-04	2.9E-03
0.8	2.2255409284	2.2255405064	4.2E-07	4.1E-04	8.1E-03
0.9	2.4596031111	2.4596027182	3.9E-07	4.8E-04	2.1E-02

Table 2: Maximum error analysis of Example 2

	Exact solution	HeWM (k=1,M=6)	Absolute errors of HeWM	Absolute errors in [7]	Absolute errors in [11]
0.1	1.1051709180	1.1052502851	7.9E-05	8.3E-06	9.5E-03
0.2	1.2214027581	1.2214285150	2.5E-05	1.7E-05	5.4E-03
0.3	1.3498588075	1.3498647980	5.9E-06	2.6E-05	4.4E-03
0.4	1.4918246976	1.4918263505	1.6E-06	3.6E-05	1.1E-02
0.5	1.6487212707	1.6487229585	1.6E-06	4.6E-05	2.3E-02
0.6	1.8221188003	1.8221211404	2.3E-06	5.7E-05	1.6E-02
0.7	2.0137527074	2.0137599700	7.2E-06	6.9E-05	8.4E-03
0.8	2.2255409284	2.2255685591	2.7E-05	8.2E-05	2.4E-03
0.9	2.4596031111	2.4596851998	8.2E-05	9.6E-05	1.5E-02

Table 3: Maximum absolute errors of Example 3

	Exact solution	HeWM (k = 1, M = 6)	Absolute errors of HeWM
0.1	1.9900000000	1.9900000000	0
0.2	1.9600000000	1.9600000000	0
0.3	1.9100000000	1.9100000000	2.2E-16
0.4	1.8400000000	1.8400000000	2.2E-16
0.5	1.7500000000	1.7500000000	0
0.6	1.6400000000	1.6400000000	0
0.7	1.5100000000	1.5100000000	2.2E-16
0.8	1.3600000000	1.3600000000	2.2E-16
0.9	1.1900000000	1.1900000000	2.2E-16

$$u(t) = 1 + x + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} P'_{n,m}(t) \tag{21}$$

Substituting (19)-(21) in (18), we obtain the following system of algebraic

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \Psi_{n,m}(t) - e^t = 0 \tag{22}$$

From here, Hermite wavelet coefficients are obtained. Approximate solution is obtained from (21) by substituting the values of wavelet coefficients.

Table 2 shows the comparison of solutions and maximum absolute errors of Example 2. Figure 2 shows the comparison of exact and Hermite wavelet solutions of Example 2.

Example 3:

Consider the nonlinear integral equation

$$u(t) = t^2 - \frac{t}{3}(2\sqrt{2} - 1) + 2 + \int_0^1 ts\sqrt{u(s)}ds,\ 0 \le t \le 1 \tag{23}$$

with initial conditions $u(0) = 2, u'(0) = 0$ and exact solution as $u(t) = 2 - t^2$

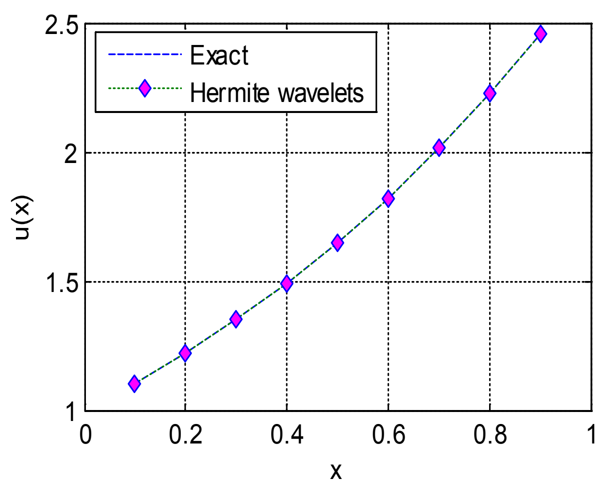

Figure 2: Comparison of exact and Hermite wavelet solution

Differentiating (23) w.r.t t, and using Leibnitz rule, it reduces to the differential equation

$$u'(t) = -2t - \frac{1}{3} - (2\sqrt{2} - 1) + 2 + \int_0^1 t\sqrt{u(t)}dt, \tag{24}$$

$$u''(t) + 2 = 0 \tag{25}$$

Consider an approximation

$$u''(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \Psi_{n,m}(t) \tag{26}$$

Integrating (26) twice and using initial conditions, we obtain

$$u'(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} P_{n,m}(t) \tag{27}$$

$$u(t) = 2 + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} P'_{n,m}(t) \tag{28}$$

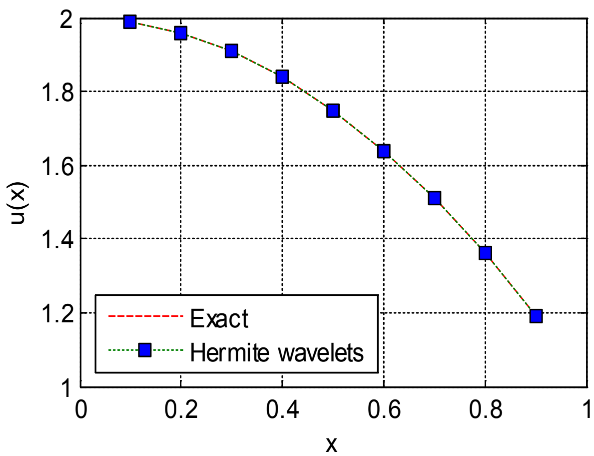

Figure 3: Comparison of exact and Hermite wavelet solution

Substituting (26)-(28) in (25), we obtain the following system of equations

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) + 2 = 0 \tag{29}$$

Solving (29) to obtain Hermite wavelet coefficients. Approximate solution is obtained from (28) by substituting the values of wavelet coefficients. Table 3 shows the comparison of exact and Hermite wavelet solutions (K = 1, M = 4) of Example 3. Figure 3 shows the comparison of exact and Hermite wavelet solutions of Example 3. Numerical results are much better in comparison to numerical reults provided in Lepik and Tamme.[12] In,[12] for $2M$ = 128, the maximum error is 3.1E-05. But, in present method, maximum error is for $2.2E - 16$, for M = 6, which is much less than errors provided in Lepik and Tamme.[12]

Conclusion

From above numerical observations, it is concluded that Hermite wavelet based numerical technique is more accurate and efficient in comparison of other numerical techniques discussed in Shiralashettiand & Mundewadi,[7] Babolian & Shahsavaran[11] and Lepik and Tamme.[12] For more accuracy, the number of collocation points may be increased.

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