Random v/s Equispaced Points for One Dimensional Monte Carlo Integration

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Abstract*:* Monte Carlo Method has been using in various fields of science, technology, research and management since a very long time. So far only random numbers have been considered for this method and research have been extended only to increase the randomness of these numbers. Instead of evaluating the function over the random points in the given range of integration by Monte Carlo Method we first divide the range of integration into n equal interval, obtain n equispaced points and then evaluate the integral over these points. Now we are interested to know that how does the choice of numbers (Random or Equispaced) affect the accuracy of one dimensional integral. Hence in this research work we are going to evaluate the one dimensional integral by Monte Carlo Method using random and equispaced points and will prove that equispaced points play a great role as far as the accuracy of one dimensional integral and pattern of decrement of error is concerned.

Keywords : Monte Carlo Method, Numerical Integration, Random Numbers, Equispaced Points, Integral Evaluation

1. INTRODUCTION

Although it may look simple at first sight to give a definition of what a random number [1] is, it proves to be quite difficult in practice. A random number is a number generated by a process, whose outcome is unpredictable, and which cannot be sub sequentially reliably reproduced. This definition works fine provided that one has some kind of a black box usually called a random number generator [2] that fulfills this task. However, if one were to be given a number, it is simply impossible to verify whether it was produced by a random number generator [3] or not. In order to study the randomness of the output of such a generator, it is hence absolutely essential to consider sequences of numbers. The Monte Carlo method [4] is a method for solving problems using random variables. It is a powerful tool in many fields of mathematics, physics and engineering [5].

Source of Random Numbers: To prove our claim unbiased we are taking the large sample of random numbers from two different sources.

Our first source is a computer program (random number generator) and by running the same we get the random numbers.We store these numbers as **".dat"** files of size 1000, 2000, 3000, 4000, 5000 with the name

Int_1_1, Int_2_1, Int_3_1, Int_1_2, Int_2_2, Int_3_2, Int_1_3, Int_2_3, Int_3_3, Int_4_1, Int_4_2, Int_4_3, Int_5_1, Int_5_2, Int_5_3,

Here the first three letters "Int" refers to the word integral, the next number refers to the file size (1000, 2000, 3000, 4000, 5000) and the last numeral refers to the no. of integral (i.e. 1^{st} or 2^{nd} or 3^{rd})

Our second source of random numbers is online generators of random numbers. For this three sides which are taken under consideration are

RANDOM.ORG (http://www.random.org/decimal-fractions)
The numbers generated by random.org are obtained as fractional values up to four decimal places between 0 and 1 and used directly in our work.

RESEARCH RANDOMIZER
(http://www.randomizer.org/form.htm)
The numbers generated by randomizer.org are obtained as integral values of four digits between 0 and 9999 and then divided by 10000 to obtain fractional values between 0 and 1 and then they are used in our work.

GRAPH PAD SOFTWARE
(http://www.graphpad.com/quickcalcs/randomn1.cfm)
The numbers generated by graphpad.com are obtained as integral values of four digits between 0 and 9999 and divided by 10000 to obtain fractional values between 0 and 1 then they are used in our work.
Five data files of random numbers from each of the above noted sites are saved as under.

File Name	Site	Size
olrr1.dat	Research Randomizer	1000
olrr1.dat	Research Randomizer	2000
olrr1.dat	Research Randomizer	3000
olrr1.dat	Research Randomizer	4000
olrr1.dat	Research Randomizer	5000
olrorg1.dat	Random.Org	1000
olrorg2.dat	Random.Org	2000
olrorg3.dat	Random.Org	3000
olrorg4.dat	Random.Org	4000
olrorg5.dat	Random.Org	5000
olgp1.dat	Graph Pad	1000
olgp2.dat	Graph Pad	2000
olgp3.dat	Graph Pad	3000
olgp4.dat	Graph Pad	4000
olgp5.dat	Graph Pad	5000

Table 1

As far as the notation and nomenclature of these files are concerned it should be noted,
ol stands for ONLINE , gp stands for GRAPH PAD , rorg stands for RANDOM.ORG , rr stands for RESEARCH RANDOMIZER and the last numeral n stands for file size multiplied by 1000.
Since it is very cumbersome to present all these numbers and program in this research paper therefore these may be seen and accessed from the web address:

1. **http://www.4shared.com/folder/BAytR7eW/data_files.html**
2. **http://www.4shared.com/office/1fGAqWsdba/random_vs_equispaced-program**

All the random numbers in the above noted files are distinct and have no correlation with each other. Before using these numbers in the research paper these numbers have gone through four methods to test [6] their independence and these methods are Poker Test [7], Run Test ([8],[9]), Frequency Test [10] and Frequency Monobit Tes [11].

2. MONTE CARLO METHOD FOR NUMERICAL INTEGRATION

Just to embrace a wide range of roblem solving techniques which use random numbers and statistics of probability, the term "Monte Carlo Method" [12] is used. The term Monte Carlo is being coined after the casino in the principality of Monte Carlo. Any method that uses random numbers to examine some problem is a Monte Carlo Method. The term Monte Carlo Method ([13], [14]) was first used by Stanslaw Ulam for simulations in Physics and other fields that require solutions for problems that are impossible to solve by traditional analytical or numerical methods. It is indeed an artificial sampling method which can be used for solving complicated problems in analytic formulation and for simulating purely statistical problems. The method is being used more and more in recent years, especially in those cases where the number of factors included in the problem is so large that an analytical solution is impossible.

The advantages of the method are, above everything is that even very difficult problems can often be treated quite easily and desired modifications can be applied without too much trouble. The poor precision is the main disadvantage of the method and as such large number of trials is necessary. This drawback is of little importance as the calculations are almost exclusively performed on automatic computers.

As far as the "Hit or Miss" [15] method is concerned, we suppose that the function f(x) is bounded by $0 \le f(x) \le c$ for $a \le x \le b$

and we wish to evaluate $\mathrm{I} = \int_a^b f(x)\,dx$

For it, we consider the following two regions

$$S = \{(x, y) : a \le x \le b; 0 \le y \le f(x)\} \qquad \& \qquad \Omega = \{(x, y) : a \le x \le b; 0 \le y \le c\}$$

i.e. S is the region below the curve of the function $f(x)$ And Ω is a rectangle that covers S completely.If |S| stands for the area S then I = |S|

The area Ω is easy to find, which is simply

$$c(b-a) \ \Omega = \{(x, y) : a \le x \le b; 0 \le y \le c\}$$

If by applying any method we are in position to find the proportion $\frac{|S|}{|\Omega|}$ Then $\mathrm{I} = |\Omega| \times \frac{|S|}{|\Omega|}$

The proportion $\frac{|S|}{|\Omega|}$ can be estimated using simulation by generating random points uniformly distributed in the region Ω and then counting how many of them are falling in S. If n be the total number of random points generated and n_s be the number of points in S then the integral I can be estimated as

$$\mathrm{I} = c(b-a)\frac{n_s}{n}$$

We can also view the problem of evaluating the integral $\mathrm{I} = \int_a^b f(x)\,dx$

Where $f(x)$ is supposed to be a continuous and real valued function on $[a,b)$ by defining a second function *g(x)* as below

$$g(x) = \begin{Bmatrix} \frac{1}{(b-a)} & when a \le x \le b \\ 0 & otherwise \end{Bmatrix}$$

On account of this definition we follow $\int_{-\infty}^{+\infty} g(x)\,dx = 1.$

Showing that we can consider *g(x)* to be the probability density function. If we insert this function *g(x)* in the constitution of I, we agree to say

$$\mathrm{I}=(b-a)\int_a^b f(x)g(x)dx=(b-a)\mathrm{E}\{f(x)\}.$$

Thus the integral I is simply the width of the integral multiplied by the expectation of the integrand. Now the very first step of Monte Carlo Problem is to set up a simulation to approximate the value of $E\{f(x)\}$

This expectation can be evaluated by taking the mean of the functional value at n randomly selected points emanating from a sequence x_n of pseudo random numbers generated by some PRN generator [16] which provides values in the interval $a \le x \le b$ .Hence I can be approximated by

$$\mathrm{I}=(b-a)\int_a^b f(x)g(x)dx=(b-a)\mathrm{E}\{f(x)\}\approx\frac{(b-a)}{n}\sum_i^n f(x_i).$$

The error term can be approximated by the root mean square deviation σ of the expectation which is given by

$$\sigma=\sqrt{\frac{f(x)^2-\mathrm{E}^2\{f(x)\}}{n}}$$

Where $\langle f(x)\rangle$ is the sequence of the functional values and $\mathrm{E}^2\{f(x)\}$ is the sequence with same size having elements equal to $\mathrm{E}^2\{f(x)\}$

3. INTEGRAL EVALUATION

In the present work, three types of one dimensional integrals [17] are taken into consideration and are evaluated by Monte Carlo method using RANDOM POINTS (online generated) as well as EQUISPACED POINTS.

First Integral (1-D)

1. The first integral under investigation is

$$\mathrm{I}_1=\int_1^2\frac{1}{(x+1)\sqrt{(x^2-1)}}dx$$

Exact value of which is 0.57735

In order to evaluate our first integral by Monte Carlo Integration ([17],[18],[19]) using both the random nodes & equispaced nodes, we developed & used a computer program.

For the data files INT_1_1.DAT; INT_1_2.DAT; INT_1_3.DAT; INT_1_4.DAT; INT_1_5.DAT With their file codes (for convenience) as “1, 7, 13, 19, 25” the error in the value of the integral is given in **table 2**

Integral-1

SELF GENERATED DATA FILES (CODES : 1,7,13,19,25)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	0.5816259	0.5611041	0.57735	-0.0042759	0.0162459
2	2000	0.5761381	0.5658818	0.57735	0.0012119	0.0114682
3	3000	0.5626247	0.5679831	0.57735	0.0147253	0.0093669
4	4000	0.5753193	0.5692395	0.57735	0.0020307	0.0081105
5	5000	0.5568186	0.5700048	0.57735	0.0205314	0.0073452

Table 2

For the data files Olgp1.DAT; Olgp2.DAT; Olgp3.DAT; Olgp4.DAT; Olgp5.DAT With their file codes (for convenience) as "6, 12, 18, 24, 30" the error in the value of the integral is given in **table 3**

Integral-1

ONLINE GENERATED DATA FILES (CODES : 6,12,18,24,30)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	0.5395241	0.5611041	0.57735	0.0378259	0.0162459
2	2000	0.5643761	0.5658818	0.57735	0.0129739	0.0114682
3	3000	0.5603055	0.5679831	0.57735	0.0170445	0.0093669
4	4000	0.5814791	0.5692395	0.57735	-0.0041291	0.0081105
5	5000	0.5726949	0.5700048	0.57735	0.0046551	0.0073452

Table 3

For the data files Olrorg1.DAT; Olrorg2.DAT; Olrorg3.DAT; Olrorg4.DAT; Olrorg5.DAT With their file codes (for convenience) as (5, 11, 17, 23, 29) the error in the value of the integral is given in **table 4**

Integral-1

ONLINE GENERATED DATA FILES (CODES : 5,11,17,23,29)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	0.5697237	0.5611041	0.57735	0.0076263	0.0162459
2	2000	0.5662909	0.5658818	0.57735	0.0110591	0.0114682
3	3000	0.5582555	0.5679831	0.57735	0.0190945	0.0093669
4	4000	0.5772865	0.5692395	0.57735	6.35E-05	0.0081105
5	5000	0.5565216	0.5700048	0.57735	0.0208284	0.0073452

Table 4

For the data files Olrr1.DAT; Olrr2.DAT; Olrr3.DAT; Olrr4.DAT; Olrr5.DAT With their file codes (for convenience) as (4, 10, 16, 22, 28) the error in the value of the integral is given in **table 5**

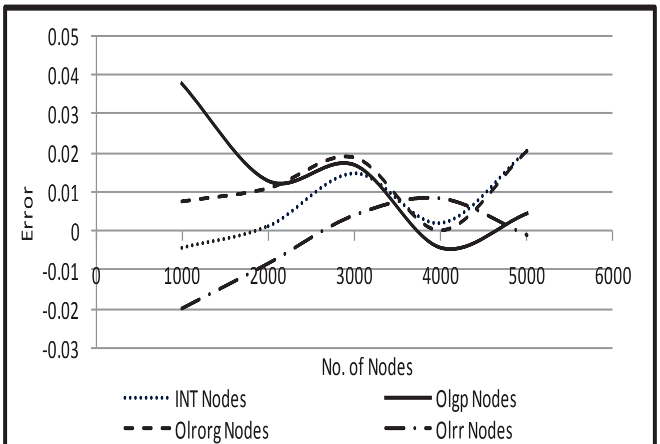
Integral-1

ONLINE GENERATED DATA FILES (CODES : 4,10,16,22,28)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	0.5974008	0.5611041	0.57735	-0.0200508	0.0162459
2	2000	0.5856733	0.5658818	0.57735	-0.0083233	0.0114682
3	3000	0.5731009	0.5679831	0.57735	0.0042491	0.0093669
4	4000	0.5687571	0.5692395	0.57735	0.0085929	0.0081105
5	5000	0.5783199	0.5700048	0.57735	-0.0009699	0.0073452

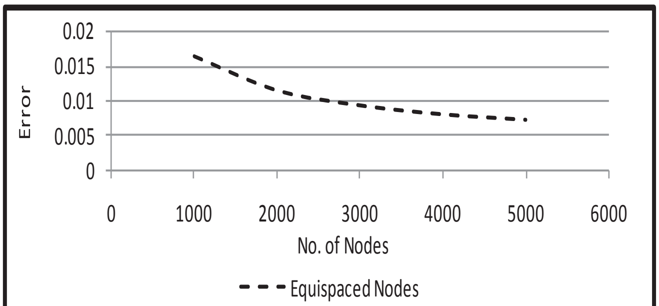
Table 5

The graphical display of error in the values of our first integral corresponding to the random nodes from table 2, 3, 4, 5 is shown in **graph 1**



Graph 1

While the display corresponding to equidistant nodes is shown in **graph 2**



Graph 2

Second Integral (1-D)

2. The second integral under investigation is

$$\mathrm{I}_2 = \int_0^1 e^x \left\{ \frac{x^3 - x + 2}{\left(x^2+1\right)^2} \right\} dx.$$

Exact value of which is 1.71828

For the evaluation of our second integral, we shall use same program with only a change in functional form, limits and the storage of number from data file as per the limits

For the data files INT_2_1.DAT; INT_2_2.DAT; INT_2_3.DAT; INT_2_4.DAT; INT_2_5.DAT With their file codes (for convenience) as "2, 8, 14, 20, 26" the error in the the value of the integral is given in **table6**

Integral-2

SELF GENERATED DATA FILES (CODES : 2,8,14,20,26)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	1.722216	1.717964	1.71828	-0.003936	0.00032
2	2000	1.713001	1.71812	1.71828	0.005279	0.00016
3	3000	1.716769	1.718178	1.71828	0.001511	0.00010
4	4000	1.712492	1.718204	1.71828	0.005788	0.00008
5	5000	1.713783	1.718221	1.71828	0.004497	0.00006

Table 6

For the data files Olgp1.DAT; Olgp2.DAT; Olgp3.DAT; Olgp4.DAT; Olgp5.DAT With their file codes (for convenience) as "6,12,18,24,30" the error in the value of the integral is given in **table7**

Integral-2

ONLINE GENERATED DATA FILES (CODES : 6,12,18,24,30)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	1.70642	1.717964	1.71828	0.01186	0.00032
2	2000	1.713795	1.71812	1.71828	0.004485	0.00016
3	3000	1.714958	1.718178	1.71828	0.003322	0.00010
4	4000	1.715862	1.718204	1.71828	0.002418	0.00008
5	5000	1.718467	1.718221	1.71828	-0.000187	0.00006

Table 7

For the data files Olrorg1.DAT; Olrorg2.DAT; Olrorg3.DAT; Olrorg4.DAT; Olrorg5.DAT with their file codes (for convenience) as "5, 11, 17, 23, 29" the error in the value of the integral is given in **table 8**

Integral-2

ONLINE GENERATED DATA FILES (CODES : 5,11,17,23,29)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	1.710681	1.717964	1.71828	0.007599	0.00032
2	2000	1.716692	1.71812	1.71828	0.001588	0.00016
3	3000	1.713194	1.718178	1.71828	0.005086	0.00010
4	4000	1.719102	1.718204	1.71828	-0.000822	0.00008
5	5000	1.718221	1.718221	1.71828	5.9E-05	0.00006

Table 8

For the data files Olrr1.DAT; Olrr2.DAT; Olrr3.DAT; Olrr4.DAT; Olrr5.DAT with their file codes (for convenience) as (4, 10, 16, 22, 28) the error in the value of the integral is given in **table 9**

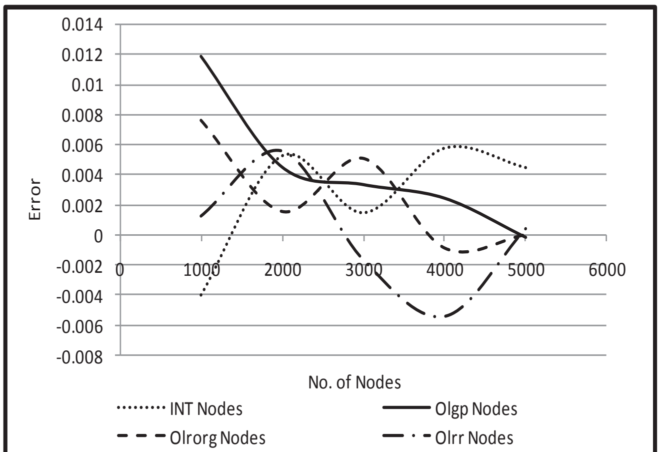
Integral-2

ONLINE GENERATED DATA FILES (CODES : 4,10,16,22,28)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	1.717001	1.717964	1.71828	0.001279	0.00032
2	2000	1.712658	1.71812	1.71828	0.005622	0.00016
3	3000	1.719927	1.718178	1.71828	-0.001647	0.00010
4	4000	1.723698	1.718204	1.71828	-0.005418	0.00008
5	5000	1.717832	1.718221	1.71828	0.000448	0.00006

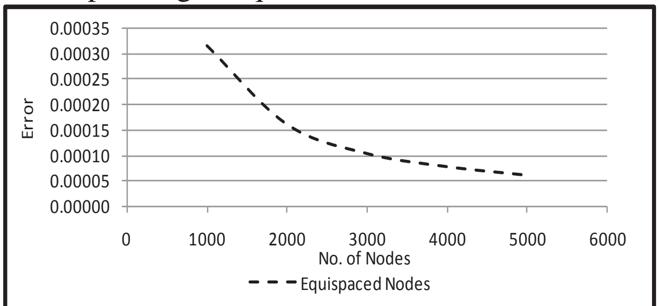
Table 9

The graphical display of error in the values of our second integral corresponding to the random nodes from table 6, 7, 8, 9 is shown in **graph 3**



Graph 3

While the display corresponding to equidistant nodes is shown in **Graph 4**



Graph 4

Third Integral (1-D)

3. The third and last integral is

$$I_3 = \int_0^1 \frac{1}{1+3e^x+2e^{2x}}\,dx$$

Exact value of which is 0.0933494.

For the evaluation of our second integral we shall use the same program with only a change in functional form, limits and the storage of number from data file as per the limits.

For the data files INT_3_1.DAT; INT_3_2.DAT; INT_3_3.DAT; INT_3_4.DAT; INT_3_5.DAT With their file codes (for convenience) as "3, 9, 15, 21, 27" the error in the value of the integral is given in **table 10**

Integral-3

SELF GENERATED DATA FILES (CODES : 3,9,15,21,27)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	0.093233	0.093287	0.093349	0.000117	0.000062
2	2000	0.092760	0.093318	0.093349	0.000589	0.000031
3	3000	0.092307	0.093329	0.093349	0.001043	0.000020
4	4000	0.092621	0.093334	0.093349	0.000728	0.000015
5	5000	0.093584	0.093338	0.093349	-0.000234	0.000012

Table 10

For the data files Olgp1.DAT; Olgp2.DAT; Olgp3.DAT; Olgp4.DAT; Olgp5.DAT with their file codes (for convenience) as “6,12,18,24,30” the error in the value of the integral is given in **table 11**

Integral-3

ONLINE GENERATED DATA FILES (CODES : 6,12,18,24,30)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	0.091553	0.093287	0.093349	0.001796	0.000062
2	2000	0.092680	0.093318	0.093349	0.000669	0.000031
3	3000	0.092996	0.093329	0.093349	0.000354	0.000020
4	4000	0.093015	0.093334	0.093349	0.000334	0.000015
5	5000	0.093449	0.093338	0.093349	-0.000099	0.000012

Table 11

For the data files Olrorg1.DAT; Olrorg2.DAT; Olrorg3.DAT; Olrorg4.DAT; Olrorg5.DAT with their file codes (for convenience) as “5, 11, 17, 23, 29” the error in the value of the integral is given in **table 12**

Integral-3

ONLINE GENERATED DATA FILES (CODES : 5,11,17,23,29)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	0.092872	0.093287	0.093349	0.000478	0.000062
2	2000	0.093296	0.093318	0.093349	0.000053	0.000031
3	3000	0.092671	0.093329	0.093349	0.000679	0.000020
4	4000	0.093421	0.093334	0.093349	-0.000072	0.000015
5	5000	0.093150	0.093338	0.093349	0.000200	0.000012

Table 12

For the data files Olrr1.DAT; Olrr2.DAT; Olrr3.DAT; Olrr4.DAT; Olrr5.DAT with their file codes (for convenience) as “4, 10, 16, 22, 28” the error in the value of the integral is given in **table 13**

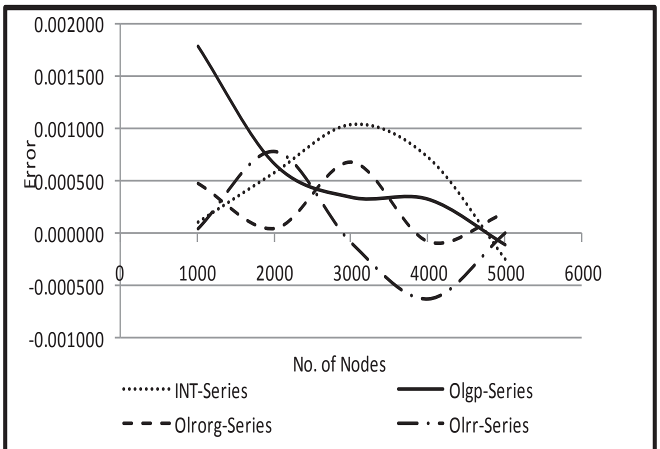
Integral-3

ONLINE GENERATED DATA FILES (CODES : 4,10,16,22,28)

S.No.	No. Of Nodes	Value Of Integral Using Random Nodes	Value Of Integral Using Equispaced Nodes	True Value	Error Using Random Nodes	Error Using Equispaced Nodes
1	1000	0.093302	0.093287	0.093349	0.000048	0.000062
2	2000	0.092565	0.093318	0.093349	0.000784	0.000031
3	3000	0.093440	0.093329	0.093349	-0.000091	0.000020
4	4000	0.093972	0.093334	0.093349	-0.000623	0.000015
5	5000	0.093343	0.093338	0.093349	0.000006	0.000012

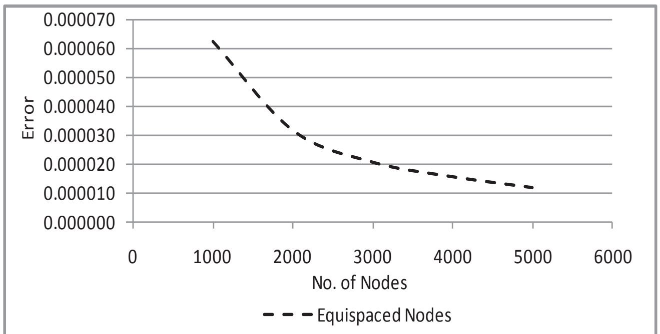
Table 13

The graphical display of error in the value of our third integral corresponding to random nodes from table 10,11,12,13 is shown in **graph 5**.



Graph 5

While the display corresponding to equidistant nodes is shown in **graph 6**



Graph 6

4. CONCLUSIONS

Here we observe that Error in the value of all the single integrals doesn't follow any pattern corresponding to different size of random numbers and also seems to be random in nature whereas using equispaced nodes the error follow a pattern and steadily approaches to zero [see graph] and true value of integral is almost achieved using 5000 equispaced points.

Hence we can conclude the following points

However random the numbers (True or Pseudo) are used in Monte Carlo integration it is not necessary that the set of random numbers which gives the best approximation (minimum error) of one integral (single) will also yield the same accuracy in the evaluation of other integral whereas if we use equispaced numbers in a given range of integration then we get almost smooth curve corresponding to the error in the values of integral having a regular decrement.

❖ It is also not necessary that corresponding to different number of random numbers obtained through same source will give the value of integral more and more approximated i.e. increase in the random numbers doesn't give the assurance for regular decrement in error whereas using equispaced nodes we get a regular decrement in the error of value of integral.

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