Semi Compatibility Mapping of Fuzzy Metric Space

Mamta Singh[1] and Amit Kumar Govery[2]

[1]*Department of Mathematical Science and Computer Application, Bundelkhand University, Jhansi (U.P.), India*
[2]*Department of Mathematics, MIT Mandsaur (M.P.), India*

Email: amitgovery@gmail.com

Abstract: Fuzzy metric space is first defined by Kramosil and Michalek. Singh B. and Chauhan were first introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. In this paper, a fixed point theorem for six self-mappings is presented by using the concept of semi compatible maps which are the generalized result.

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1. INTRODUCTION

The concept of Fuzzy sets was initially investigated by Zadeh [7] as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined Fuzzy metric space in various ways. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek [8] and modified by George and Veeramani [1]. Recently, Grebiec [8] has proved fixed-point results for Fuzzy metric space. In the sequel, Singh and Chauhan [3] introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Jain and singh[2] proved a fixed point theorem for six self maps in a fuzzy metric space.

In this paper, a fixed point theorem for six self maps has been established using the concept of semi compatibility of pairs of self maps in fuzzy metrics space, which generalizes the result of Cho [9].

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2. PRELIMINARIES

Definition 2.1. [10] A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is called a *t-norm* if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for a, b, c, d$\in$[0, 1].

Examples of t-norms are $a*b=ab$ *and* $a*b=\min\{a,b\}$.

Definition 2.2. [10] The 3-tuple (X, M, *) is said to be a *Fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions: For all $x,y,z\in X$ and $s,t>0$

(FM-1) M (x, y, 0) = 0,

(FM-2) M (x, y, t) =1 for all t > 0 if and only if x = y,

(FM-3) M (x, y, t) = M (y, x, t),

(FM-4) M (x, y, t) * M (y, z, s) $\le$ M (x, z, t + s),

(FM-5) M (x, y, $\cdot$): [0,∞) $\to$ [0, 1] is left continuous,

(FM-6) $\lim_{t\to\infty}$ M (x, y, t) =1.

Note that M (x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M (x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [10] Let (X, d) be a metric space. $Difine\ a*b=\min\{a,b\}\ and\ M(x,y,t)=\dfrac{t}{t+d(x,y)}$ for all $x,y\in X$ and $t>0$.
Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [10] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a *Cauchy sequence* if and only if for each $\varepsilon>0$, t > 0, there exists $n_0\in N$ such that M (X_n, X_m, t) > 1 - ε for all n, m $\ge n_0$. The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each $\varepsilon>0$, t > 0 there exists $n_0\in$ N such that M (x_n, x, t) > 1 - ε for all n,m $\ge n_0$. A Fuzzy metric space (X, M, *) is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [3] Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be *compatible* if and only if M (ASx_n, SAx_n ,t) $\to$ 1 for all t > 0, whenever $\{x_n\}$ is a sequence in X such that Sx_n, $Ax_n \to$ p for some p in X as $n \to \infty$.

Definition 2.5 [10] Suppose A and S be two maps from a Fuzzy metric space (X,M,*) into itself. Then they are said to be semi- compatible if $ASx_n = Sx$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} SAx_n = x \in X$. It follows that (A, S) is semi- compatible and $A_y = S_y$ imply $AS_y = SA_y$ by taking $\{x_n\} = y$ and $x = A_y = S_y$.

Proposition 2.1. [2] In a fuzzy metric space (X, M, *) limit of a sequence is unique.

Lemma 2.1. [8] Let (X, M, *) be a fuzzy metric space. Then for all x, y $\in$ X, M (x, y, .) is a non-decreasing function.

Lemma 2.2. [9] Let (X, M, *) be a fuzzy metric space. If there exists k $\in$ (0, 1) such that for all x, y $\in$ X, M (x, y, kt) $\geq$ M (x, y, t) $\forall$ t > 0, then x = y.

Lemma 2.3. [2] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number k $\in$(0, 1) such that M (x_{n+2}, x_{n+1}, kt) $\geq$M (x_{n+1}, x_n, t). t > 0 and n $\in$N. Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.4. [4] The only t-norm * satisfying r * r $\geq$ r for all r $\in$[0, 1] is the minimum t-norm, that is a * b = min {a, b} for all a, b $\in$ [0, 1].

3. MAIN RESULT

Theorem 3.1. Let (X, M, *) be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:
(a) P (X) $\subset$ ST (X), Q(X) $\subset$ AB(X);
(b) AB = BA, ST = TS, PB = BP, QT = TQ;
(c) Either AB or P is continuous;
(d) (P, AB) and (Q, ST) pairs are semi compatible;
(e) There exists q $\in$ (0, 1) such that for every x, y $\in$ X and t > 0
M (Px, Qy, qt) $\geq$ M (ABx, STy, t) * M (Px, ABx, t) * M (Qy, STy,t) * M (Px, STy, t).
Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: Let $x_0 \in$ X. From (a) there exist x_1, $x_2 \in$ X such that $Px_0 = STx_1$ and $Qx_1 = ABx_2$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Px_{2n-2} = STx_{2n-1} = y_{2n-1}$ and $Qx_{2n-1} = ABx_{2n} = y_{2n}$ for n = 1, 2, 3, ...

Step 1. Put $x = x_{2n}$ and $y = x_{2n+1}$ in (e), we get

$$M(Px_{2n}, Qx_{2n+1}, qt) \geq M(ABx_{2n}, STx_{2n+1}, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qx_{2n+1}, STx_{2n+1}, t) * M(Px_{2n}, STx_{2n+1}, t).$$

$$= M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+1}, t)\}$$

$$\geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t).$$

From lemma 2.1 and 2.2, we have

$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t)$.

Similarly,

$M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t)$.

Thus,

$M(y_{n+1}, y_{n+2}, qt) \geq M(y_n, y_{n+1}, t)$ for $n = 1, 2, \ldots$

$$M(y_n, y_{n+1}, t) \geq M(y_n, y_{n+1}, t/q) \geq M(y_{n-2}, y_{n-1}, t/q^2) \ldots \ldots \ldots \geq M(y_1, y_2, t/q^n) \to 1 \text{ as } n \to \infty,$$

and hence $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$ for any $t > 0$. For each $\varepsilon > 0$ and $t > 0$, we can choose $n_0 \in N$ such that $M(y_n, y_{n+1}, t) > 1 - \varepsilon$ for all $n > n_0$. For $m, n \in N$, we suppose $m \geq n$. Then we have

$$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, t/m-n) * M(y_{n+1}, y_{n+2}, t/m-n) * \ldots * M(y_{m-1}, y_m, t/m-n)$$

$$\geq (1 - \varepsilon) * (1 - \varepsilon) * \ldots * (1 - \varepsilon)\ (m - n) \text{ times}$$

$\geq (1 - \varepsilon)$ and hence $\{y_n\}$ is a Cauchy sequence in X. Since (X, M, *) is complete, $\{y_n\}$ converges to some point $z \in X$. Also its subsequences converges to the same point $z \in X$ i.e.,

$\{Qx_{2n+1}\} \to z$; $\{STx_{2n+1}\} \to z$ (1)

and $\{Px_{2n}\} \to z$; $\{ABx_{2n}\} \to z$. (2)

Case I. Suppose AB is continuous.

Since AB is continuous, we have $(AB)^2x_{2n} \to ABz$ and $ABPx_{2n} \to ABz$. As (P, AB) is semi compatible pair, then $PABx_{2n} \to ABz$.

Step 2. Put $x = ABx_{2n}$ and $y = x_{2n+1}$ in (e), we get

$$M(PABx_{2n}, Qx_{2n+1}, qt) \geq M(ABABx_{2n}, STx_{2n+1}, t) * M(PABx_{2n}, ABABx_{2n}, t) * M(Qx_{2n+1}, STx_{2n+1}, t) * M(PABx_{2n}, STx_{2n+1}, t).$$

Taking $n \to \infty$, we get

$$M(ABz, z, qt) \geq M(ABz, z, t) * M(ABz, ABz, t) * M(z, z, t) * M(ABz, z, t)$$

$\geq M(ABz, z, t) * M(ABz, z, t)$ i.e. $M(ABz, z, qt) \geq M(ABz, z, t)$.

Therefore, by using lemma 2.2, we get $ABz = z$ (3)

Step 3. Put $x = z$ and $y = x_{2n+1}$ in (e), we have

$$M(Pz, Qx_{2n+1}, qt) \geq M(ABz, STx_{2n+1}, t) * M(Pz, ABz, t) * M(Qx_{2n+1}, STx_{2n+1}, t) * M(Pz, STx_{2n+1}, t).$$

Taking $n \to \infty$ and using equation (1), we get

$$M(Pz, z, qt) \geq M(z, z, t) * M(Pz, z, t) * M(z, z, t) * M(Pz, z, t)$$
$$\geq M(Pz, z, t) * M(Pz, z, t) \text{ i.e. } M(Pz, z, qt) \geq M(Pz, z, t).$$

Therefore, by using lemma 2.2, we get $Pz = z$. Therefore, $ABz = Pz = z$.

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ in condition (e), we get

$$M(PBz, Qx_{2n+1}, qt) \geq M(ABBz, STx_{2n+1}, t) * M(PBz, ABBz, t) * M(Qx_{2n+1}, STx_{2n+1}, t) * M(PBz, STx_{2n+1}, t).$$

As $BP = PB$, $AB = BA$, so we have $P(Bz) = B(Pz) = Bz$ and $(AB)(Bz) = (BA)(Bz) = B(ABz) = Bz$. Taking $n \to \infty$ and using (1), we get

$$M(Bz, z, qt) \geq M(Bz, z, t) * M(Bz, Bz, t) * M(z, z, t) * M(Bz, z, t)$$
$$\geq M(Bz, z, t) * M(Bz, z, t) \text{ i.e. } M(Bz, z, qt) \geq M(Bz, z, t).$$

Therefore, by using lemma 2.2, we get $Bz = z$ and also we have $ABz = z$. $Az = z$.

Therefore, $Az = Bz = Pz = z$ (4)

Step 5. As $P(X) \subset ST(X)$, there exists $u \in X$ such that $z = Pz = STu$. Putting $x = x_{2n}$ and $y = u$ in (e), we get

$$M(Px_{2n}, Qu, qt) \geq M(ABx_{2n}, STu, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qu, STu, t) * M(Px_{2n}, STu, t).$$

Taking $n \to \infty$ and using (1) and (2), we get

$$M(z, Qu, qt) \geq M(z, z, t) * M(z, z, t) * M(Qu, z, t) * M(z, z, t)$$
$$\geq M(Qu, z, t) \text{ i.e. } M(z, Qu, qt) \geq M(z, Qu, t).$$

Therefore, by using lemma 2.2, we get $Qu = z$. Hence $STu = z = Qu$. Since (Q, ST) is semi compatible therefore, we have $QSTu = STQu$. Thus $Qz = STz$.

Step 6. Putting $x = x_{2n}$ and $y = z$ in (e), we get

$$M(Px_{2n}, Qz, qt) \geq M(ABx_{2n}, STz, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qz, STz, t) * M(Px_{2n}, STz, t).$$

Taking $n \to \infty$ and using (2) and step 5, we get

$$M(z, Qz, qt) \geq M(z, Qz, t) * M(z, z, t) * M(Qz, Qz, t) * M(z, Qz, t)$$
$$\geq M(z, Qz, t) * M(z, Qz, t) \text{ i.e. } M(z, Qz, qt) \geq M(z, Qz, t).$$

Therefore, by using lemma 2.2, we get $Qz = z$.

Step 7. Putting $x = x_{2n}$ and $y = Tz$ in (e), we get

$$M(Px_{2n}, QTz, qt) \geq M(ABx_{2n}, STTz, t) * M(Px_{2n}, ABx_{2n}, t) * M(QTz, STTz, t) * M(Px_{2n}, STTz, t).$$

As $QT = TQ$ and $ST = TS$, we have $QTz = TQz = Tz$ and $ST(Tz) = T(STz) = TQz = Tz$.
Taking $n \to \infty$, we get

$$M(z, Tz, qt) \geq M(z, Tz, t) * M(z, z, t) * M(Tz, Tz, t) * M(z, Tz, t)$$
$$\geq M(z, Tz, t) * M(z, Tz, t) \text{ i.e. } M(z, Tz, qt) \geq M(z, Tz, t).$$

Therefore, by using lemma 2.2, we get $Tz = z$. Now $STz = Tz = z$ implies $Sz = z$.
Hence $Sz = Tz = Qz = z$ (5)

Combining (4) and (5), we get $Az = Bz = Pz = Qz = Tz = Sz = z$. Hence, z is the common fixed point of A, B, S, T, P and Q.

Case II. Suppose P is continuous. As P is continuous, $P^2x_{2n} \to Pz$ and $P(AB)x_{2n} \to Pz$.
As (P, AB) is semi compatible, we have $(AB)Px_{2n} \to Pz$.

Step 8. Putting $x = Px_{2n}$ and $y = x_{2n+1}$ in condition (e), we have

$$M(PPx_{2n}, Qx_{2n+1}, qt) \geq M(ABPx_{2n}, STx_{2n+1}, t) * M(PPx_{2n}, ABPx_{2n}, t)$$
$$* M(Qx_{2n+1}, STx_{2n+1}, t) * M(PPx_{2n}, STx_{2n+1}, t).$$

Taking $n \to \infty$, we get

$$M(Pz, z, qt) \geq M(Pz, z, t) * M(Pz, Pz, t) * M(z, z, t) * M(Pz, z, t)$$
$$\geq M(Pz, z, t) * M(Pz, z, t) \text{ i.e. } M(Pz, z, qt) \geq M(Pz, z, t).$$

Therefore by using lemma 2.2, we have $Pz = z$.
Further, using steps 5, 6, 7, we get $Qz = STz = Sz = Tz = z$.

Step 9. As $Q(X) \subset AB(X)$, there exists $w \in X$ such that $z = Qz = ABw$.
Put $x = w$ and $y = x_{2n+1}$ in (e), we have

$$M(Pw, Qx_{2n+1}, qt) \geq M(ABw, STx_{2n+1}, t) * M(Pw, ABw, t)$$
$$* M(Qx_{2n+1}, STx_{2n+1}, t) * M(Pw, STx_{2n+1}, t).$$

Taking $n \to \infty$, we get

$$M(Pw, z, qt) \geq M(z, z, t) * M(Pw, z, t) * M(z, z, t) * M(Pw, z, t)$$
$$\geq M(Pw, z, t) * M(Pw, z, t) \text{ i.e. } M(Pw, z, qt) \geq M(Pw, z, t).$$

Therefore, by using lemma 2.2, we get $Pw = z$. Therefore, $ABw = Pw = z$. As (P, AB) is semi compatible, we have $Pz = ABz$. Also, from step 4, we get $Bz = z$. Thus, $Az = Bz = Pz = z$ and we see that z is the common fixed point of the six maps in this case also.

Uniqueness: Let u be another common fixed point of A, B, S, T, P and Q.

Then $Au = Bu = Pu = Qu = Su = Tu = u$. Put $x = z$ and $y = u$ in (e), we get

$$M(Pz, Qu, qt) \geq M(ABz, STu, t) * M(Pz, ABz, t) * M(Qu, STu,.t) * M(Pz, STu, t).$$

Taking n $\to \infty$, we get

$M(z, u, qt) \geq M(z, u, t) * M(z, z, t) * M(u, u, t) * M(z, u, t)$

$\geq M(z, u, t) * M(z, u, t)$ i.e. $M(z, u, qt) \geq M(z, u, t)$.

Therefore by using lemma 2.2, we get z = u. Therefore z is the unique common fixed point of self-maps A, B, S, T, P and Q.

Remark 3.1. If we take B = T = I then condition (b) of theorem 3.1, is satisfied trivially .

Corollary 3.1. Let (X, M, *) be a complete fuzzy metric space and let A, S, P and Q be mappings from X into itself such that the following conditions are satisfied:

(a) $P(X) \subset S(X)$, $Q(X) \subset A(X)$;

(b) either A or P is continuous;

(c) (P, A) and (Q, S) pairs are semi compatible;

(d) there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$

$M(Px, Qy, qt) \geq M(Ax, Sy, t) * M(Px, Ax, t) * M(Qy, Sy, t) * M(Px, Sy, t)$. Then A, S, P and Q have a unique common fixed point in X.

Remark 3.2. In view of remark 3.1, corollary 3.1 is a generalization of the result of Cho [9] in the sense that condition of compatibility of the pairs of self-maps has been restricted to semi compatibility and only one map of the first pair is needed to be continuous.

4. CONCLUSION

In this paper, a fixed point theorem for six self-mappings is presented by using the concept of semi compatibility which is the generalized result.

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References:

[1] A. George and P. Veeramani, *on some results in Fuzzy metric spaces*. Fuzzy Sets and Systems 64 (1994), 395-399.

[2] A. Jain and B. Singh, *A fixed point theorem using weak compatible in fuzzy metric space,* Varahmihir J. Math. Sci.Vol .5 No. I (2005), 297-306.

[3] B. Singh and M.S. Chouhan, *Common fixed points of compatible maps in Fuzzy metric spaces,* Fuzzy sets and systems, 115 (2000), 471-475.

[4] E.P. Klement, R. Mesiar and E. Pap, *Triangular Norms,* Kluwer Academic Publishers.

[5] G. Jungck, P.P. Murthy and Y.J. Cho, *Compatible mappings of type (A) and common fixed points,* Math. Japonica, 38 (1993), 381-390.

[6] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica 11 (1975), 336-344.

[7] L. A. Zadeh, *Fuzzy sets,* Inform and control 89 (1965), 338-353.

[8] M. Grabiec, *Fixed points in Fuzzy metric space,* Fuzzy sets and systems, 27(1998), 385-389.

[9] S.H., Cho, *on common fixed point theorems in fuzzy metric spaces*, J. Appl. Math. & Computing Vol. 20 (2006), No. 1 -2, 523-533.

[10] S.N. Mishra, N. Mishra and S.L. Singh, *Common fixed point of maps in fuzzy metric space*, Int. J. Math. Math. Sci. 17(1994), 253-258.

[11] Y.J. Cho, *Fixed point in Fuzzy metric space,* J. Fuzzy Math. 5(1997), 949-962.