

Nonlinear Retarded differential Equations with Nonlocal History Conditions

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ABSTRACT

In this paper we prove the existence and uniqueness of a strong solution of nonlinear retarded differential equation with a nonlocal history condition using the method of semidiscretization in time. Some applications of the abstract results is given in last section.

Keywords

Nonlinear Retarded differential Equation, Method of lines, Strong solution.

Mathematics Subject Classification (2000): 34G20, 34K30, 47H06

1 Introduction: Let us suppose the following nonlinear retarded differential equation in a real Hilbert space H ,

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), u(r(t))), t \in (0, T], \\ h(u|_{[-\tau, 0]}) = \phi_0, \text{ on } [-\tau, 0], \end{cases} \quad (1.1)$$

where $0 < \tau, T < \infty$, $\phi_0 \in C_0 = C([- \tau, 0]; H)$, the nonlinear operator A is single-valued and maximal monotone defined from the domain $D(A) \subset H$ into H , the nonlinear map f is defined from $[0, T] \times H^2$ into H , the map h is defined from C_0 into C_0 and $\psi|_{[-\tau, 0]}$ is the restriction of $\psi \in C_T := C([- \tau, T]; H)$. Here $C_t := C([- \tau, t]; H)$ for $t \in [0, T]$ is the Banach space of all continuous functions from $[-\tau, t]$ into H endowed with the supremum norm

$$\|\phi\|_t = \sup_{-\tau \leq \eta \leq t} \|\phi(\eta)\|, \phi \in C_t,$$

where $\|\cdot\|$ represents the norms in H and the function $r: [0, T] \rightarrow [-\tau, T]$.

We can also applied the existence and uniqueness results for (1.1) to the particular case, namely, the retarded functional differential equation,

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), u(t - \tau)), t \in (0, T], \\ u = \phi_0, \text{ on } [-\tau, 0], \end{cases} \quad (1.2)$$

Byszewski and Lakshmikantham [1], Byszewski [2], Balachandran and Chandrasekaran [3], Lin and Liu [4] and the references cited in these papers, establish the existence, uniqueness and stability of different types of solutions of differential and functional differential equations with nonlocal conditions. We aim to extend the application of the method of lines to (1.1). For the applications of the method of lines to nonlinear evolution and nonlinear functional evolutions, we refer to Kartsatos and Parrott [5, 6], Kartsatos [7], Bahuguna and Raghavendra [8], Bahuguna [9], and the references cited in these papers.

Suppose that there is $\chi \in C_T$ such that $h(\chi|_{[-\tau, 0]}) = \phi_0$ on $[-\tau, 0]$ and $\chi(0) \in D(A)$. We prove the existence of a strong solution u of (1.1) under the assumptions of Theorem(2.1), stated in the next section, in the sense that there exists a unique function $u \in C_T$ such that $u(t) \in D(A)$ for a.e. $t \in [0, T]$, u is differentiable a.e. on $[0, T]$ and

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), u(r(t))), a.e. t \in [0, T] \\ (u_{[-\tau, 0]}) = (\chi_{[-\tau, 0]}), \text{ on } [-\tau, 0]. \end{cases} \quad (1.3)$$

Finally, we show that u is unique if and only if $\chi \in C_T$ satisfying $h(\chi_{[-\tau, 0]}) = \emptyset_0$ is unique up to $[-\tau, 0]$.

2 Preliminaries and Main result

Let H be a real Hilbert space. Let $\langle x, y \rangle$ be the inner product of $x, y \in H$. Further, we assume the following assumptions:

(A1) The operator $A: D(A) \subset H \rightarrow H$ is maximal monotone, i.e.,

$\langle Ax - Ay, x - y \rangle \geq 0$, for all $x, y \in D(A)$ and $R(I + A) = H$, where $R(\cdot)$ is the range of an operator.

(A2) The map $h: C_0 \rightarrow C_0$ and there exists $\chi \in C_T$ such that $h(\chi_{[-\tau, 0]}) = \emptyset_0$ and $\chi(0) \in D(A)$.

(A3) The nonlinear map $f: [0, T] \times H^2 \rightarrow H$ satisfies a local Lipschitz-like condition

$$\|f(t, u_1, u_2) - f(s, v_1, v_2)\| \leq L_f(R)[|t-s| + \|u_1 - v_1\| + \|u_2 - v_2\|],$$

For all (u_1, u_2) and (v_1, v_2) in $B_R(H^2, (\chi(0), \chi(0)))$ and $t \in [0, T]$ where $L_f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non decreasing function and, for $R > 0$,

$$B_R(H^2, (\chi(0), \chi(0))) = \{(u_1, u_2) \in H^2: \sum_{i=1}^2 \|u_i - \chi(0)\| \leq R\}.$$

(A4) The maps $r: [0, T] \rightarrow [-\tau, T]$ are continuous satisfying the delay property $r(t) \leq t$ for $t \in [0, T]$.

Theorem 2.1 Suppose that the assumptions (A1)-(A4) are satisfied. Then (1.1) has a strong solution $u \in C_T$ either on $[-\tau, T]$ or on the maximal interval of existence $[-\tau, t_{\max})$, $0 < t_{\max} \leq T$, and in the later case either $\lim_{t \rightarrow t_{\max}^-} \|u(t)\| = \infty$ or $u(t)$ goes to the boundary of $D(A)$ as $t \rightarrow t_{\max}^-$. Moreover, u is Lipschitz continuous on every compact subinterval of existence.

3 Discretization Scheme and a Priori Estimates

In this part we prove the existence and uniqueness of a strong solution to (1.3) for any given $\chi \in C_T$ with $\chi(0) \in D(A)$. For the application of the method of lines to (1.3), we proceed as follows. Let $R_0 := \sup_{t \in [-\tau, T]} \|\chi(t) - \chi(0)\|$. For any $0 < R \leq R_0$, we set t_0 such that $0 < t_0 \leq T$, $t_0[\|A\chi(0)\| + 3L_f(R_0)(T + (m+1)R_0) + \|f(0, \chi(0), \chi(0))\|] \leq R$.

For $n \in \mathbb{N}$, let $h_n = t_0/n$. We set $u_0^n = \chi(0)$ for all $n \in \mathbb{N}$ and define each of $\{u_j^n\}_{j=1}^n$ as the unique solution of the equation

$$\frac{u - u_{j-1}^n}{h_n} + Au = f\left(t_j^n, u_{j-1}^n, \check{u}_{j-1}^n(r(t_j^n))\right), \quad (3.1)$$

where $\check{u}_0^n(t) = \chi(t)$ for $t \in [-\tau, 0]$, $\check{u}_0^n(t) = \chi(0)$ for $t \in [0, t_0]$ and for $2 \leq j \leq n$

$$\check{u}_{j-1}^n(t) = \begin{cases} \chi(t) & \text{for } t \in [-\tau, 0] \\ u_{i-1}^n + \frac{1}{h_n}(t - t_{i-1}^n)(u_i^n - u_{i-1}^n) & \text{for } t \in (t_{i-1}^n, t_i^n) i = 1, 2, \dots, j-1 \\ u_{j-1}^n & \text{for } t \in [t_{j-1}^n, t_0] \end{cases} \quad (3.2)$$

The existence of a unique $u_j^n \in D(A)$ satisfying (3.1) is a consequence of the m -accretivity of A . Using (A2) we first prove that the points $\{u_j^n\}_{j=0}^n$ lie in a ball with its radius independent of the discretization parameters j , h_n and n . We then prove a priori estimates on the difference quotients $\left\{\frac{u_j^n - u_{j-1}^n}{h_n}\right\}$ using (A2). We define the sequence $\{U^n\} \subset C_T$ of polygonal functions

$$U^n(t) = \begin{cases} \chi(t) & \text{for } t \in [-\tau, 0] \\ u_{j-1}^n + \frac{1}{h_n}(t - t_{j-1}^n)(u_j^n - u_{j-1}^n) & \text{for } t \in (t_{j-1}^n, t_j^n) \end{cases} \quad (3.3)$$

and prove the convergence of $\{U^n\}$ to a unique strong solution u of (1.3) in C_T as $n \rightarrow \infty$.

Now, we first show that $\{u_j^n\}_{j=0}^n$ lie in a ball in H of radius independent of j , h_n and n .

Lemma 3.1 For $n \in \mathbb{N}$, $j = 1, 2, \dots, n$,

$$\|u_j^n - \chi(0)\| \leq R.$$

Proof: From (3.1) for $j = 1$ and the accretivity of A , we have

$$\|u_1^n - \chi(0)\| \leq h_n [\|A\chi(0)\| + 3L_f(R_0)(T + 2R_0) + \|f(0, \chi(0), \chi(0))\|] \leq R.$$

Assume that $\|u_i^n - \chi(0)\| \leq R$ for $i = 1, 2, \dots, j-1$. Now for $2 \leq j \leq n$,

$$\|u_j^n - \chi(0)\| \leq \|u_{j-1}^n - \chi(0)\| + h_n [\|A\chi(0)\| + 3L_f(R_0)(T + 2R_0) + \|f(0, \chi(0), \chi(0))\|].$$

Repeating the above inequality, we get

$$\|u_j^n - \chi(0)\| \leq j h_n [\|A\chi(0)\| + 3L_f(R_0)(T + 2R_0) + \|f(0, \chi(0), \chi(0))\|] \leq R.$$

as $j h_n \leq t_0$ for $0 \leq j \leq n$. This completes the proof of the lemma.

Now we establish a *priori* estimates for the difference quotients $\left\{\frac{u_j^n - u_{j-1}^n}{h_n}\right\}$.

Lemma 3.2 There exists a positive constant K independent of the discretization parameters n, j and h_n such that

$$\left\|\frac{u_j^n - u_{j-1}^n}{h_n}\right\| \leq K, j = 1, 2, \dots, n, n = 1, 2, \dots$$

Proof: In this proof and subsequently, K will represent a generic constant independent of j , h_n and n .

Subtracting $Au_0^n = A\chi(0)$ from both the sides in (3.1) and applying $F(u_1^n - u_0^n)$, using accretivity of A , we get

$$\left\|\frac{u_1^n - u_0^n}{h_n}\right\| \leq [\|A\chi(0)\| + 3L_f(R_0)(T + 2R_0) + \|f(h_n, \chi(0), \chi(0))\|] \leq K.$$

Now, for $2 \leq j \leq n$ applying $F(u_j^n - u_{j-1}^n)$, to (3.1) and using accretivity of A , we get

$$\left\|\frac{u_j^n - u_{j-1}^n}{h_n}\right\| \leq \left\|\frac{u_{j-1}^n - u_{j-2}^n}{h_n}\right\| + \left\|f\left(t_j^n, u_{j-1}^n, \tilde{u}_{j-1}^n(r(t_j^n))\right) - f\left(t_{j-1}^n, u_{j-2}^n, \tilde{u}_{j-2}^n(r(t_{j-1}^n))\right)\right\|.$$

From the above inequality we get

$$\max_{\{1 \leq k \leq j\}} \left\| \frac{u_k^n - u_{k-1}^n}{h_n} \right\| \leq (1 + Ch_n) \max_{\{1 \leq k \leq j-1\}} \left\| \frac{u_k^n - u_{k-1}^n}{h_n} \right\| + Ch_n,$$

where C is a positive constant independent of j, h_n and n . Proceeding the above inequality, we get

$$\max_{\{1 \leq k \leq j\}} \left\| \frac{u_j^n - u_{j-1}^n}{h_n} \right\| \leq ((1 + Ch_n)^j \cdot jCh_n) \leq TCe^{TC} \leq K.$$

Thus we find the proof of the lemma.

We consider another sequence $\{X^n\}$ of step functions from $[0, T]$ into H by

$$X^n(t) = \begin{cases} \chi(0) & \text{for } t = 0, \\ u_j^n & \text{for } t \in (t_{j-1}^n, t_j^n) \end{cases}$$

Remark 3.3 From the lemma 3.2 it follows that the functions U^n and $\widetilde{u}_r^n, 0 \leq r \leq n-1$, are Lipschitz continuous on $[0, t_0]$ with a uniform Lipschitz constant K . The sequence $U^n(t) - X^n(t) \rightarrow 0$ in H as $n \rightarrow \infty$ uniformly on $[-\tau, t_0]$. Furthermore, $X^n(t) \in D(A)$ for $t \in [0, t_0]$ and the sequences $\{U^n(t)\}$ and $\{X^n(t)\}$ are bounded in H , uniformly in $n \in \mathbb{N}$ and $t \in [-\tau, t_0]$. The sequence $\{AX^n(t)\}$ is bounded uniformly in $n \in \mathbb{N}$ and $t \in [0, t_0]$.

For convenience, let

$$f^n(t) = f\left(t_j^n, u_{j-1}^n, \widetilde{u}_{j-1}^n\left(r(t_j^n)\right)\right), t \in (t_{j-1}^n, t_j^n), 1 \leq j \leq n.$$

Then equation (3.1) may be rewritten as

$$\frac{d^-}{dt} U^n(t) + AX^n(t) = f^n(t), t \in (0, t_0] \quad (3.4)$$

where $\frac{d^-}{dt}$ denotes the left derivative in $(0, t_0]$. Also, for $t \in (0, t_0]$, we have

$$\int_0^t AX^n(s)ds = \chi(0) - U^n(t) + \int_0^t f^n(s)ds. \quad (3.5)$$

Lemma 3.4 There exists $u \in C_{t_0}$ such that $U^n \rightarrow u$ in C_{t_0} as $n \rightarrow \infty$. Moreover, u is Lipschitz continuous on $[0, t_0]$.

Proof: From (3.4) for $t \in (0, t_0]$, we have

$$\left\langle \frac{d^-}{dt} (U^n(t) - U^k(t)), X^n(t) - X^k(t) \right\rangle \leq \langle f^n(t) - f^k(t), X^n(t) - X^k(t) \rangle.$$

In view of the above inequality, we find

$$\begin{aligned} & \frac{1}{2} \frac{d^-}{dt} \|U^n(t) - U^k(t)\|^2 \\ & \leq \left\langle \frac{d^-}{dt} (U^n(t) - U^k(t)) - f^n(t) + f^k(t), U^n(t) - U^k(t) - X^n(t) + X^k(t) \right\rangle + \langle f^n(t) - f^k(t), U^n(t) - U^k(t) \rangle. \end{aligned}$$

Now,

$$\|f^n(t) - f^k(t)\| \leq \epsilon_{nk}^m(t) + K\|U^n - U^k\|_t,$$

where $\epsilon_{nk}(t) = K[h_n + h_k + \|X^n(t - h_n) - U^n(t)\| + \|X^k(t - h_k) - U^k(t)\| + \{|r(t) - r(t_j^n)| + |r(t) - r(t_i^k)|\}]$,

for $t \in (t_{j-1}^n, t_j^n]$ and $t \in (t_{l-1}^k, t_l^k]$, $1 \leq j \leq n$, $1 \leq l \leq k$. Therefore $\epsilon_{nk}(t) \rightarrow 0$ as $n, k \rightarrow \infty$ uniformly on $[0, t_0]$,

$$\frac{d^-}{dt} \|U^n(t) - U^k(t)\|^2 \leq K[\epsilon_{nk} + \|U^n - U^k\|_t^2],$$

where ϵ_{nk} is a sequence of numbers such that $\epsilon_{nk} \rightarrow 0$ as $n, k \rightarrow \infty$. Integrating the above inequality over $(0, s)$, $0 < s \leq t \leq t_0$, taking the supremum over $(0, t)$ and using the fact that $U^n = \emptyset$ on $[-\tau, 0]$ for all n , we get

$$\|U^n - U^k\|_t^2 \leq K \left[T\epsilon_{nk} + \int_0^t \|U^n - U^k\|_s^2 ds \right].$$

Using Gronwall's inequality we summarize that there exists $u \in C_{t_0}$ such that $U^n \rightarrow u$ in C_{t_0} . explicitly, $u = \emptyset$ on $[-\tau, 0]$ and from remark 3.3 it follows that u is Lipschitz continuous on $[0, t_0]$. Hence we get the proof of the lemma.

Proof of theorem 2.1: Firstly we prove the existence on $[-\tau, t_0]$ and then prove the unique continuation of the solution on $[-\tau, T]$. Proceeding similarly, we may show that $u(t) \in D(A)$ for $t \in [0, t_0]$, $AX^n(t) \rightharpoonup Au(t)$ on $[0, t_0]$ and $Au(t)$ is weakly continuous on $[0, t_0]$. Here $\rightharpoonup$ denotes the weak convergence in H . For every $x^* \in X^*$ and $t \in (0, t_0]$, we have

$$\int_0^t \{AX^n(s), x^*\} ds = \{ \chi(0), x^* \} - \{U^n(t), x^*\} + \int_0^t \{f^n(s), x^*\} ds.$$

Applying lemma 3.4 and the bounded convergence theorem, we get as $n \rightarrow \infty$,

$$\int_0^t \{Au(s), x^*\} ds = \{ \chi(0), x^* \} - \{u(t), x^*\} + \int_0^t \{f(s, u(s), u(r(s))), x^*\} ds. \quad (3.6)$$

Since $Au(t)$ is Bochner integrable [10] on $[0, t_0]$, from (3.6) we get

$$\frac{d}{dt} u(t) + Au(t) = f(t, u(t), u(r(t))), \text{ a.e. } t \in [0, t_0]. \quad (3.7)$$

Clearly, u is Lipschitz continuous on $[0, t_0]$ and $u(t) \in D(A)$ for $t \in [0, t_0]$. Now we prove the uniqueness of a function $u \in C_{t_0}$ which is differentiable a.e. on $[0, t_0]$ with $u(t) \in D(A)$ a.e. on $[0, t_0]$ and $u = \emptyset$ on $[-\tau, 0]$ satisfying (3.7). Let $u_1, u_2 \in C_{t_0}$ be two such functions. Assume that

$$R = \max\{\|u_1\|_{t_0}, \|u_2\|_{t_0}\}.$$

Then for $u = u_1 - u_2$, we have

$$\frac{d}{dt} \|u(t)\|^2 \leq C(R) \|u\|_t^2, \text{ a.e. } t \in [0, t_0],$$

where $C: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing function. Integrating over $(0, s)$ for $0 < s \leq t \leq t_0$, taking supremum over $(0, t)$ and using the fact that $u \equiv 0$ on $[-\tau, 0]$, we get

$$\|u\|_t^2 \leq C(R) \int_0^t \|u\|_s^2 ds.$$

Application of Gronwall's inequality implies that $u \equiv 0$ on $[-\tau, t_0]$.

Now we prove the continuation of the solution u on $[-\tau, T]$. Suppose $t_0 < T$ and consider the problem

$$\begin{cases} w'(t) + Aw(t) = \tilde{f}(t, w(t), w(\tilde{r}(t))), & 0 < t \leq T - t_0, \\ w = \tilde{\chi}, & \text{on } [-\tau - t_0, 0], \end{cases} \quad (3.8)$$

where $\tilde{f}(t, u_1, u_2) = f(t + t_0, u_1, u_2)$, $0 < t \leq T - t_0$,

$$\tilde{\chi}(t) = \begin{cases} \chi(t + t_0), & t \in [-\tau - t_0, -t_0], \\ u(t + t_0), & t \in [-t_0, 0], \end{cases}$$

$$\tilde{r}(t) = r(t + t_0) - t_0, \quad t \in [0, T - t_0].$$

Since $\tilde{\chi}(0) = u(t_0) \in D(A)$ and $\tilde{f}$ satisfies (A2) and $\tilde{r}$ satisfies (A4) on $[0, T - t_0]$, we may follow as before and prove the existence of a unique $w \in C([-\tau - t_0, t_1]; X)$, $0 < t_1 \leq T - t_0$, such that w is a Lipschitz continuous on $[0, t_1]$, $w(t) \in D(A)$ for $t \in [0, t_1]$ and w satisfies

$$\begin{cases} w'(t) + Aw(t) = \tilde{f}(t, w(t), w(\tilde{r}(t))), & a.e. t \in [0, t_1], \\ w = \tilde{\chi}, & \text{on } [-\tau - t_0, 0], \end{cases} \quad (3.9)$$

Then the function $\bar{u}(t) = \begin{cases} u(t), & t \in [-\tau, t_0], \\ w(t - t_0), & t \in [t_0, t_0 + t_1], \end{cases}$

is Lipschitz continuous on $[0, t_0 + t_1]$, $\bar{u}(t) \in D(A)$ for $t \in [0, t_0 + t_1]$ and satisfies (1.3) a.e. on $[0, t_0 + t_1]$. Proceeding in the same manner we may prove the existence either on the whole interval $[-\tau, T]$ or on the maximal interval of existence $[-\tau, t_{max})$, $0 < t_{max} \leq T$. In case $\lim_{t \rightarrow t_{max}} \|u(t)\| < \infty$, then as $u(t) \in D(A)$ for $t \in [0, t_{max})$, we have $\lim_{t \rightarrow t_{max}} u(t)$ is in the closure of $D(A)$ in H and if it is in $D(A)$ then we can widen $u(t)$ beyond t_{max} contradicting the definition of the maximal interval of existence.

Now, let u_x be the strong solution of (1.3) corresponding to χ satisfying $h(\chi_{[-\tau, 0]}) = \emptyset_0$. If there are $\chi^1, \chi^2 \in C_T$ ($\tilde{T}$ is either equal to T or $\tilde{T} < t_{max}$) such that $h(\chi^1_{[-\tau, 0]}) = h(\chi^2_{[-\tau, 0]}) = \emptyset_0$ and $\chi^1 \neq \chi^2$ on $[-\tau, 0]$ then clearly u_{χ^1} and u_{χ^2} satisfying (1.3) are different. Hence the proof of the theorem(2.1).

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